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LETTER TO THE EDITOR

Caustics and energy flux in general relativity

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Abstract. A useful coordinate system in the vicinity of a caustic point is described. The local expression of the Hamilton–Jacobi phase function of the geodesic flow and the first order WKB approximation of Einstein's equations in a neighbourhood of the caustic are given.

Bondi *et al* (1962) calculated the energy loss of gravitationally radiating isolated sources as measured at infinity in an asymptotically flat space–time. Their expression for the energy loss, namely, the integrated square modulus of the rate of change of the shear, is also encountered on studying the propagation of discontinuities of the metric tensor (Dautcourt 1969). Similarly, this expression appears in the first-order WKB approximation of Einstein's equations. Note that in these last two cases there is no need to assume that the distance from the sources is large. All the mentioned calculations were performed in the so called Bondi (or null) coordinates, which may be introduced only if neighbouring null geodesics do not cross each other. In other words, the expression that has been found by Bondi and which describes, in the first-order WKB approximation, the total energy flux of gravitational radiation is valid only far from the caustics. At the caustic we are at a loss. In general relativity caustics are quite common (Hawking and Ellis 1973). In fact, they appear just where matter is, or in its vicinity. Moreover, experimental detection of gravitational radiation is generally done close to the sources and the measurement is affected by the laboratory and its environment (Pisarev 1976). Therefore it is desirable to evaluate at least an approximate expression for the total energy flux of gravitational radiation in the vicinity of the caustic and not only far from it. The key problem is to find a coordinate system in a neighbourhood of a caustic point.

In this letter we describe such a coordinate system. We also give the local expression of the Hamilton–Jacobi phase function of the geodesic flow and the first-order WKB approximation of Einstein's equations in the neighbourhood of the caustic. This can be used, following Bondi *et al* (1962) and Penrose (1967), to express the total energy flux of gravitational radiation near a caustic point in the limit where the field oscillates rapidly.

It will be convenient to describe the geodesic and its tangent 'simultaneously', or, in mathematical terms, to work in the cotangent bundle rather than in space–time. Furthermore, in general relativity we are interested in the following situation. There is a space-like sub-manifold N of space–time and we look at the focal points of geodesics orthogonal to this sub-manifold. In the case where N is three dimensional the geodesics are time-like, while in the case when it is two dimensional they are chosen to be null. All our considerations are valid away from the singularities of space–time and it is always assumed that the tangent to a geodesic never vanishes. The points of the caustic may be of two different types. If x is a point of the cotangent bundle such that its projection on space–time is a caustic point then one can find coordinates $\{x^\lambda, p_\lambda\}$, $0 \leqslant \lambda \leqslant 3$, in a neighbourhood of x such that the constant affine parameter sections of the geodesic flow are described either by

$$x^0 = p_0^2, \qquad p_k = \text{constant}, \qquad 1 \leqslant k \leqslant 3 \tag{1}$$

or by

$$x^j = p_0 \cdot p_j, \qquad 3-d+1 \leqslant j \leqslant 3, \qquad p_r = \text{constant}, \qquad 0 \leqslant r \leqslant 3-d \tag{2}$$

where d is the number of linearly independent Jacobi fields vanishing at the projection of x on space–time.

Using the theory of Lagrangian manifolds (Maslov 1965†) it is not difficult to exhibit, after some simple coordinate transformations of (2), the following convenient form for the local expression of the Hamilton–Jacobi phase function of the geodesic flow (more precisely, a canonically transformed phase function). In case (1)

$$S(p_0, x^1, x^2, x^3) = 3^{-1} p_0^3 - p_0 x^0 - p_1 x^1 - p_2 x^2 - p_3 x^3 \tag{3}$$

and in case (2),

$$\begin{aligned} S(x^0, \ldots, x^{3-d}, p_{3-d+1}, \ldots, p_3) &= \{2^{-1} p_0 \cdot p_{3-d+1}{}^2 - 2^{-1} p_0^3 + p_0(x^{3-d+1} - d^{-1} x^0) - p_{3-d+1} x^{3-d+1}\} + \ldots \\ &\quad + \{2^{-1} p_0 \cdot p_3^2 - 2^{-1} p_0^3 + p_0(x^3 - d^{-1} x^0) - p_3 x^3\} - p_1 x^1 - \ldots p_{3-d} x^{3-d}. \end{aligned} \tag{4}$$

Notice that these expressions resemble two of the 'potential functions' of Thom's elementary catastrophes (Thom 1973): the first is a fold and the second is a 'multiple' cut of an elliptic umbilic. We thus encounter, not unexpectedly, a connection between the focal points of the geodesic flow and Thom's theory of elementary catastrophes.

It can be shown, using some results from the theory of stability of mappings (Guillemin and Schaeffer 1973), that in the short-wavelength approximation the metric, in the vicinity of a caustic point, can be written as follows:

$$\begin{aligned} g_{\mu\nu}(x, \omega) = g_{(0)\mu\nu}(x) + \omega^{-1} \exp(\mathrm{i}\omega\phi) \Big(& a_{\mu\nu}(x) A(\omega^{2/3}\theta_1, \ldots, \omega^{2/3}\theta_k) \\ & \times \sum_{j=1}^{k} b^j_{\mu\nu}(x)\, \omega^{-1/3} \frac{\partial A}{\partial \theta_j}(\omega^{2/3}\theta_1, \ldots, \omega^{2/3}\theta_k) \Big) + \mathrm{O}(\omega^{-2}) \end{aligned} \tag{5}$$

in which ω is a large positive parameter (in linear theories it is the frequency, yet regarding ω as the frequency in Einstein's theory is frequently a misnomer); $g_{(0)\mu\nu}$ is a hyperbolic normal metric (cf Choquet-Bruhat 1969), for an expression valid far from the caustic), A is a universal entire function and $\phi, \theta_1, \ldots, \theta_k$ are arbitrary smooth functions on space–time. The integer k is 1 in the case of a fold point and 2 in the case of an elliptic umbilic. If $k = 1$ then A is the well known Airy function (of negative argument), while if $k = 2$ it is a natural generalization of the Airy function.

On putting (5) into Einstein's equations we get (in addition to some exceptional cases that we shall not consider here) a certain combination of $\phi, \theta_1, \ldots, \theta_k$ which defines a smooth null hypersurface across which radiation flows. The tensors $a_{\mu\nu}$ and $b^j_{\mu\nu}$ satisfy equations closely analogous to those found on studying the propagation of discontinuities of the metric tensor (Treder 1962).

† Translated into French by J Lascoux and R Seneor and published by Dunod, Paris in 1972.

The Bondi-type expression for the approximate total energy flux near the caustics can then be written down.

More about this will appear elsewhere.

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